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WEAK CONVERGENCE OF COMPOUND STOCHASTIC PROCESS, I*

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Abstract: Compound stochastic processes are constructed by taking the superpositive of independent copies of secondary processes, each of which is initiated at an epoch of a renewal process called the primary process. Suppose there are M possible k-dimensional secondary processes $\{\xi^\nu(t): t \geq 0\}$, $\nu = 1, 2, ..., M$. At each epoch of the renewal process $\{A(t): t \geq 0\}$ we initiate a random number of each of the M types. Let $\{m_l: l \geq 1\}$ be a sequence of M-dimensional random vectors whose components specify the number of secondary processes of each type initiated at the various epochs. The compound process we study is

$$Y(t) = \sum_{l=1}^{A(t)} \sum_{\nu=1}^{M} \sum_{j=1}^{M_{l\nu}} \xi_{lj}^{\nu}(t-\tau_l), \quad t \geq 0,$$

where the $\xi_{lj}^{\nu}(\cdot)$ are independent copies of ξ^ν, $m_{l\nu}$ is the νth component of m and $\{\tau_l: l \geq 1\}$ are the epochs of the renewal process. Our interest in this paper is to obtain functional central limit theorems for $\{Y(t): t \geq 0\}$ after appropriately scaling the time parameter and state space. A variety of applications are discussed.

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1. Introduction and summary

Compound stochastic processes (c.s.p.'s) are constructed by taking the superposition of independent copies of secondary processes, each of which is initiated at an epoch of a renewal process called the primary process. Suppose there are M possible k-dimensional secondary processes $\{\xi^\nu(t): t \geq 0\}$, $\nu = 1, 2, ..., M$. At each epoch of the renewal process $\{A(t): t \geq 0\}$, we initiate a random number of each of the M types. Let

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$\{m_l: l \geq 1\}$ be a sequence of M-dimensional random vectors whose components specify the number of secondary processes of each type initiated at the various epochs. The compound process we study is

$$(1) \qquad Y(t) = \sum_{l=1}^{A(t)} \sum_{\nu=1}^{M} \sum_{j=1}^{m_{l\nu}} \xi_{lj}^{\nu}(t-\tau_l), \quad t \geq 0,$$

where the $\xi_{lj}^{\nu}(\cdot)$ are independent copies of ξ^{ν}, $m_{l\nu}$ is the νth component of m_l, and $\{\tau_l: l \geq 1\}$ are the epochs of the renewal process. We shall frequently use superscripts in this manner and not mean powers. The context should prevent confusion. Our interest in this paper is to obtain functional central limit theorems (f.c.l.t.'s) for $\{Y(t): t \geq 0\}$ after appropriately scaling the time parameter and state space.

In this paper, we shall only rescale the time parameter of the primary process, and let the secondary processes run in their original time scale. To this end, we set $A_n(t) \equiv A(nt)$ for $0 \leq t \leq 1$ and $n = 1, 2, \ldots$. If the renewal process $\{A(t): t \geq 0\}$ has an arrival rate (the reciprocal of the expected time between epochs) of λ (which we shall always assume is positive and finite), then the renewal process $\{A_n(t): 0 \leq t \leq 1\}$ has arrival rate $n\lambda$. Hence the effect of rescaling the primary process is to greatly speed up the rate at which secondary processes are generated. Thus when we consider the c.s.p. $\{Y_n(t): 0 \leq t \leq 1\}$ with primary process $\{A_n(t): 0 \leq t \leq 1\}$, we shall be studying the case in which there is a high rate of generation of secondary processes. The limit theorems obtained as n goes to infinity can then be viewed as approximation for a c.s.p. with a high arrival rate. In a sequel to this paper, we shall consider the situation in which both the primary and secondary process time parameters are rescaled.

We shall assume that secondary processes have sample paths in the space D^k, the Cartesian product of k (≥ 1) copies of $D[0, 1] \equiv D$, the space of right-continuous functions on $[0, 1]$ having left limits. The Borel sets of D^k are denoted by $\mathcal{D}^k$, where the topology on D^k is the product topology formed from the Skorohod topology on D. The processes of interest, $\{Y_n: n \geq 1\}$, also have paths in D^k and are defined by

$$(2) \qquad Y_n(t) = \sum_{l=1}^{A_n(t)} \sum_{\nu=1}^{M} \sum_{j=1}^{m_{l\nu}} \xi_{lj}^{\nu}(t-\tau_l(n)), \quad 0 \leq t \leq 1,$$

where $\tau_l(n) = \tau_l/n$. Under appropriate moment assumptions we shall show that $n^{-1/2}(Y_n - nb)$ converges weakly to a certain k-dimensional Gaussian process, where b is a given deterministic function. Similar results can be obtained for non-homogeneous renewal processes on $[0, 1]$ by defining $\tau_l(n) = \lambda(\tau_l/n)$, where λ is a strictly increasing continuous map of $[0, 1]$ onto $[0, 1]$.

A large literature exists on c.s.p.'s for very special secondary processes; see, for example, Takács [14], Parzen [11, §4.5], Borovkov [3] and Port [12; 13]. The method used in this paper was suggested by Borovkov [3], who studied the many server queue as the number of servers becomes large. Here we shall discuss as applications an urn model with immigration, the GI/G/∞ queue, shot noise, and several other models. We note in passing that the c.s.p. includes as special cases partial sums of random variables, renewal processes, and random sums of random variables.

The organization of this paper is as follows: Section 2 deals with the construction of the probability space and a number of preliminary results. In Section 3, the convergence of finite-dimensional distributions is treated. Section 4 deals with tightness of the process $\{Y_n : n \geq 1\}$. Finally, in Section 5 we discuss applications of these results for specific secondary processes.

2. Preliminaries

The probability space that will support the basic compound processes (eq. 2) is constructed as follows. For $l = 1, 2, \ldots$, set $\Omega_l^1 = \mathrm{R}_+ \times \mathrm{N}^M$, where $\mathrm{R}_+ = [0, \infty)$ and N^M is the set of M-dimensional vectors with nonnegative components. Let $\mathcal{R}_+$ and $\mathcal{N}^M$ be the corresponding Borel sets, where the topology on R_+ is induced by the Euclidean metric and the topology on N^M is the discrete topology. Set $\mathcal{F}_l^1 = \mathcal{R}_+ \times \mathcal{N}^M$, the product Borel field (B.F.), and let P_l^1 be a probability measure (p.m.) on $\mathcal{F}_l^1$ which is identical for all $l = 1, 2, \ldots$.

Next take

$$\Omega^1 = \mathrm{X}_{l=1}^{\infty} \Omega_l^1, \quad \mathcal{F}^1 = \mathrm{X}_{l=1}^{\infty} \mathcal{F}_l^1, \quad P^1 = \mathrm{X}_{l=1}^{\infty} P_l^1$$

(the product measure). Now let

$$\Omega_{l\nu}^2 = \mathrm{X}_{j=1}^{\infty} D_j^k, \quad \mathcal{F}_{l\nu}^2 = \mathrm{X}_{j=1}^{\infty} \mathcal{D}_j^k, \quad P_{l\nu}^2 = \mathrm{X}_{j=1}^{\infty} P_{j\nu},$$

where $D_j^k = D^k$, $\mathcal{D}_j^k = \mathcal{D}^k$, and $P_{j\nu}$ is a p.m. on $\mathcal{D}^k$ which is identical for all values of $j = 1, 2, \ldots$. Let

$$\Omega_l^2 = \mathsf{X}_{\nu=1}^M \Omega_{l\nu}^2, \quad \mathcal{F}_l^2 = \mathsf{X}_{\nu=1}^M \mathcal{F}_{l\nu}^2, \quad P_l^2 = \mathsf{X}_{\nu=1}^M P_{l\nu}^2,$$

and set

$$\Omega^2 = \mathsf{X}_{l=1}^\infty \Omega_l^2, \quad \mathcal{F}^2 = \mathsf{X}_{l=1}^\infty \mathcal{F}_l^2, \quad P^2 = \mathsf{X}_{l=1}^\infty P_l^2 .$$

Finally, the large space $(\Omega, \mathcal{F}, P)$ is taken to be $(\Omega^1 \times \Omega^2, \mathcal{F}^1 \times \mathcal{F}^2, P^1 \times P^2)$.

Let $\omega = (\omega^1, \omega^2)$ be a generic point of Ω. Then $\omega^1 = (\omega_1^1, \omega_2^1, \ldots)$ with $\omega_l^1 = (\omega_{l1}^1, \omega_{l2}^1)$, where $\omega_{l1}^1 \in R_+$ and $\omega_{l2}^1 \in N^M$. On the other hand, $\omega^2 = (\omega_1^2, \omega_2^2, \ldots)$ with $\omega_l^2 = (\omega_{l1}^2, \omega_{l2}^2, \ldots, \omega_{lM}^2)$ and $\omega_{l\nu}^2 = (\omega_{l\nu 1}^2, \omega_{l\nu 2}^2, \ldots)$, where $\omega_{l\nu j}^2 \in D^k$. We define the coordinate functions $u_l(\omega) = \omega_{l1}^1$, $m_l(\omega) = \omega_{l2}^1$ and $\xi_{lj}^\nu(\omega) = \omega_{l\nu j}^2$ for $l = 1, 2, \ldots$, $\nu = 1, 2, \ldots, M$, and $j = 1, 2, \ldots$. The random variable (r.v) u_l will represent the time between the lth and $(l-1)$st epochs of the renewal process, m_l will be the M-dimensional random vector with non-negative components which specifies the number of secondary processes of each type generated at the l^{th} epoch of the renewal process, and the ξ_{lj}^ν will be independent copies of the ν^{th} type secondary process.

To sum up, we allow dependence between u_l and m_l, although the sequences $\{(u_l, m_l) : l \geqslant 1\}$ and $\{\xi_{lj}^\nu : \nu = 1, \ldots, M; l, j = 1, 2, \ldots\}$ consist of independent random elements and are independent of each other.

The renewal process $\{A(t) : t \geqslant 0\}$ is defined in the usual way in terms of the u_l's. Let $\tau_0 = 0$, $\tau_n = u_1 + \ldots + u_n$ $(n \geqslant 1)$ and define for $t \geqslant 0$

$$A(t) = \begin{cases} n & \text{on} \quad \{\tau_n \leqslant t < \tau_{n+1}\} \\ +\infty & \text{on} \quad \bigcap_{n=1}^{\infty} \{\tau_n \leqslant t\} \end{cases}$$

We shall let $\mathrm{E}\{u_l\} = \lambda^{-1}$, where $0 < \lambda < \infty$, and $A_n(t) \equiv A(nt)$ for $0 \leqslant t \leqslant 1$ and $n = 1, 2, \ldots$. Also we assume $\mathrm{E}\{u_l^2\} < \infty$.

For any vector x, with or without subscripts or superscripts, we denote the i^{th} component by x_i. We take x to be a column vector and x' the corresponding row vector. Let the $\mathrm{E}\{m_l\} = \mu$ and assume that $0 < \mu_i < \infty$ for $i = 1, 2, \ldots, M$. We assume that $\mathrm{E}\{m_{l\nu}^2\} < \infty$ and that the covariance matrix $\Sigma = \mathrm{E}\{(m_l - \mu)(m_l - \mu)'\}$ is finite and positive definite.

Next we give some easy limit theorems for the processes $\{T_n : n \geqslant 1\}$ defined by

$$T_n(t) = n^{-1} \sum_{l=1}^{A_n(t)} m_l, \quad 0 \leqslant t \leqslant 1. \tag{3}$$

Denote by ρ the metric of uniform convergence on D^k,

$$\rho(x,y) = \max\{\sup[|x_i(t) - y_i(t)| : 0 \leqslant t \leqslant 1] : 1 \leqslant t \leqslant k\},$$

where $x, y \in D^k$. We use the symbol $\Rightarrow$ to denote weak convergence. Let L_c be the linear function ct on $[0, 1]$, where c is a constant vector.

Lemma 2.1. $\lim_{n\to\infty} \rho(T_n, L_{\lambda\mu}) = 0$ *a.e.*

Proof. For a fixed $t \in [0, 1]$, $T_n(t) \to \lambda\mu t$ a.e. by the strong laws for sums of random vectors and for renewal processes. The argument for the uniform convergence is standard; cf. [8, Theorem 3.1].

Lemma 2.2. *If f is bounded and measurable then for* $0 \leqslant t \leqslant 1$,

$$\lim_{n\to\infty} \mathrm{E}\left\{\int_0^t f(\tau)\,\mathrm{d}T_{n\nu}(\tau)\right\} = \lambda\mu_\nu \int_0^t f(\tau)\,\mathrm{d}\tau$$

Proof. Since f is bounded and measurable, $T_{n\nu}(t) \to \lambda\mu_\nu t$ a.e. for all $t \in [0, 1]$, and Lebesgue measure has no atoms, we have by virtue of weak convergence (cf. [2, Theorem 5.2]),

$$\lim_{n\to\infty} \int_0^t f(\tau)\,\mathrm{d}T_{n\nu}(\tau) = \lambda\mu_\nu \int_0^t f(\tau)\,\mathrm{d}\tau \quad \text{a.e.}$$

To complete the proof, it suffices to show that

$$\sup_{n\geqslant 1} \mathrm{E}\left\{\left|\int_0^t f(\tau)\,\mathrm{d}T_{n\nu}(\tau)\right|^2\right\} = M < \infty$$

(cf. [5, Theorem 4.5.2]). Since

$$\mathrm{E}\left\{\left|\int_0^t f(\tau)\,\mathrm{d}T_{n\nu}(\tau)\right|^2\right\} \leqslant C^2\,\mathrm{E}\{T_{n\nu}^2(1)\},$$

where $|f| \leqslant C$, we need to show that the sequence $\mathrm{E}\{T_{nv}^2(1): n \geqslant 1\}$ is bounded. Recall that $\mathrm{E}\{T_{nv}^2(1)\} = \mathrm{E}\{n^{-2}(m_{1v} + \ldots + m_{A(n)v})^2\}$. Next we compute $\mathrm{E}\{[(m_{1v} - \mu_v) + \ldots + (m_{A(n)+1,v} - \mu_v)]^2\}$. For convenience, let $m_{1v} - \mu_v = x_v$. Then

$$\mathrm{E}\{(x_1 + \ldots + x_{A(n)+1})^2\} = \sum_{k=1}^{\infty} \int_{\{A(n)+1=k\}} (x_1 + \ldots + x_k)^2 \, \mathrm{d}P^1$$

$$= \sum_{k=1}^{\infty} \left\{ \sum_{i=1}^{k} \int_{\{A(n)+1=k\}} x_i^2 \, \mathrm{d}P^1 + 2 \sum_{i=1}^{k} \sum_{j=1}^{i-1} \int_{\{A(n)+1=k\}} x_i x_j \mathrm{d}P^1 \right\}$$

$$= \sum_{i=1}^{\infty} \left\{ \int_{\{A(n)+1 \geqslant i\}} x_i^2 \, \mathrm{d}P^1 + 2 \sum_{j=1}^{i-1} \int_{\{A(n)+1 \geqslant i\}} x_i x_j \mathrm{d}P^1 \right\}$$

$$= \sum_{i=1}^{\infty} \left\{ \mathrm{E}\{x_1^2\} \, \mathrm{P}[A(n) + 1 \geqslant i] + 2\mathrm{E}\{x_1\} \sum_{j=1}^{i-1} \int_{\{A(n)+1 \geqslant i\}} x_j \mathrm{d}P^1 \right\}$$

$$= \sigma^2(m_{1v}) \, \mathrm{E}\{A(n) + 1\},$$

where we have used the fact that x_i and the indicator function $\{A(n) + 1 \geqslant i\}$ are independent r.v.'s. Using the c_r-inequality we obtain

$$\mathrm{E}\{(m_1 + \ldots + m_{A(n)+1,v})^2\}$$

$$\leqslant 2\mathrm{E}\{(x_1 + \ldots + x_{A(n)+1})^2\} + 2\mu^2 \mathrm{E}\{(A(n) + 1)^2\}.$$

Since $n^{-2} \mathrm{E}\{(A(n) + 1)^2\}$ is bounded (cf. [5, p. 127]), so is $n^{-2} \mathrm{E}\{(m_1 + \ldots + m_{A(n)+1,v})^2\}$ and hence also $\mathrm{E}\{T_{nv}^2(1)\}$. This completes the proof.

As a simple corollary of this lemma we have

Corollary 2.3. *If* $\mu^v(\cdot) = \mathrm{E}\{\xi^v(\cdot)\}$ *is bounded and measurable for* $v = 1, 2, \ldots, M$, *then as* $n \to \infty$,

$$\mathrm{E}[Y_{ni}(t)] \sim n\lambda \sum_{v=1}^{M} \mu_v \int_0^t \mu_i^v(t-\tau) \mathrm{d}\tau.$$

Proof. Since

$$\mathrm{E}\{Y_{ni}(t)\} = \sum_{\nu=1}^{M} \mathrm{E}\left\{\sum_{l=1}^{A_n(t)} \sum_{j=1}^{m_{l\nu}} \xi_{lji}^{\nu}(t-\tau_l(n))\right\}$$
$$= n \sum_{\nu=1}^{M} \mathrm{E}\left\{\int_0^t \mu_i^{\nu}(t-\tau)\, \mathrm{d}T_{n\nu}(\tau)\right\},$$

a simple application of the lemma yields the result.

In a similar, but vastly more complicated manner, the above lemma can be used to obtain an asymptotic expression for cov $\{Y_{ni}(s), Y_{nj}(t)\}$. It also grows like n.

Now define the random element T_n^* of D^k as

$$(4)\qquad T_n^*(t) = n^{-1/2} \sum_{l=1}^{A_n(t)} m_l - n\lambda\boldsymbol{\mu}\, t, \quad 0 \leqslant t \leqslant 1.$$

Recall that $\boldsymbol{\Sigma} = \mathrm{E}\{(m_l - \boldsymbol{\mu})(m_l - \boldsymbol{\mu})'\}$, the covariance of m_l.

Lemma 2.4. $T_n^* \Rightarrow \lambda^{1/2}\, \boldsymbol{\Gamma}^{1/2}\, \boldsymbol{\eta}^* = \boldsymbol{\eta}$, *as* $n \to \infty$, *where*

$$\boldsymbol{\Gamma} = \boldsymbol{\Sigma} + (\lambda^2 \mathrm{E}\{u_1^2\} + 1)\boldsymbol{\mu}\boldsymbol{\mu}' - \lambda\boldsymbol{\mu}\mathrm{E}\{u_1 m_1'\} - \lambda\mathrm{E}\{u_1 m_1\}\,\boldsymbol{\mu}'$$

and $\boldsymbol{\eta}^*$ *has components that are independent, standard Brownian motions on* $[0, 1]$.

Proof. Write $T_n^*(t)$ as follows:

$$(5)\qquad T_n^*(t) = n^{-1/2} \sum_{l=1}^{A(nt)} (m_l - \lambda u_l \boldsymbol{\mu})$$
$$\times\; n^{-1/2}\lambda\boldsymbol{\mu}[nt - (u_1 + \ldots + u_{A(nt)})].$$

The random vectors $\{m_l - \lambda u_l \boldsymbol{\mu}\colon l \geqslant 1\}$ are independent, identically distributed (i.i.d.) by construction and have mean 0, covariance $\boldsymbol{\Gamma}$. By the vector version of the f.c.l.t. for random sums (cf. [7, Theorem 9]), the first term of (5) converges weakly to η. The second term of (5) converges weakly to 0 (cf. [8, Theorem 4.1]). The proof is completed by the "Converging Together Theorem" ([2, Theorem 4.1]).

Finally, for the sake of convenience we state two results that will be required later. A proof of the first lemma due to Liggett and Rosén can be found in [9, Lemma 3.2], and of the second in [7, Theorem 5]. Here, $C \equiv C[0, 1]$ and

$$w_X(\delta) = \sup\{|X(s) - X(t)|: |s-t| < \delta\},\ 0 < \delta \leqslant 1.$$

Lemma 2.5. *Let* $\{X_n\}$ *be a sequence of random functions in* $(D, \mathcal{D})$, *and* X *a random function such that* $\mathbf{P}\{X \in C\} = 1$. *If* $X_n \Rightarrow X$, *then* $\{X_n\}$ *is C-tight: for all positive* ϵ *and* η, *there exists* δ $(0 < \delta < 1)$ *and integer* n_0 *such that*

$$\mathbf{P}\{w_{X_n}(\delta) \geqslant \epsilon\} \leqslant \eta \quad \textit{for} \quad n \geqslant n_0.$$

Lemma 2.6. *Let* $\{X_n\}$ *and* X *be random functions on* $(D^k, \mathcal{D}^k)$. *Then* (i) *and* (ii) *are necessary and sufficient conditions for* $X_n \Rightarrow X$:

(i) $(X_n(t_1), ..., X_n(t_k)) \Rightarrow (X(t_1), ..., X(t_k))$ *whenever* $t_1, ..., t_k \in T_X$, *where* $T_X = \{t \in [0, 1]: \mathbf{P}[X(t) \neq X(t-)] = 0\}$.

(ii) *The sequence of marginal functions* $\{X_{ni}\}$, $i = 1, ..., k$, *on* $(D, \mathcal{D})$ *are tight.*

3. Convergence of finite-dimensional distributions

Our goal in this section is to prove that the f.d.d.'s of the processes $\{Y_n : n \geqslant 1\}$ converge weakly, under appropriate moment assumptions, to those of a certain vector-valued Gaussian process.

We begin by introducing some notation for the moments of the secondary processes. The time points, s, t, u shall always satisfy $0 \leqslant s \leqslant t \leqslant u \leqslant 1$. For $\nu = 1, 2, ..., M$, and $l, j = 1, 2, ...$, let

$$\boldsymbol{\mu}^\nu(t) = \mathbf{E}\{\boldsymbol{\xi}_{lj}^\nu(t)\},$$

$$\overset{\nu}{\boldsymbol{\Sigma}}(s, t) = \mathbf{E}\{[\boldsymbol{\xi}_{lj}^\nu(s) - \boldsymbol{\mu}^\nu(s)]\ [\boldsymbol{\xi}_{lj}^\nu(t) - \boldsymbol{\mu}^\nu(t)]'\},$$

$$\zeta_i^\nu(t) = \mathbf{E}\{|\xi_{lji}^\nu(t)|^3\},\quad i = 1, 2, ..., k.$$

We shall assume that μ_i^ν is a real-valued function of bounded variation on $[0, 1]$, and that $\sigma_{ij}^\nu(\cdot, \cdot)$ is a bounded measurable function for each i, j and ν.

Next define the vector-valued function h on $[0, 1]$ as

$$h_i(t) = \lambda \sum_{\nu=1}^{M} \mu_\nu \int_0^t \mu_i^\nu(\tau)\,\mathrm{d}\tau, \quad 0 \leqslant t \leqslant 1, \quad i = 1, 2, \ldots, k,$$

and the sequence of processes $\{X_n : n \geqslant 1\}$ in D^k as

$$X_{ni}(t) = n^{-1/2}\,(Y_{ni}(t) - nh_i(t)), \quad 0 \leqslant t \leqslant 1, \quad i = 1, 2, \ldots, k.$$

Let $\{X^1(t): 0 \leqslant t \leqslant 1\}$ be a k-dimensional centered Gaussian process with covariance function

$$\Gamma^1(s, t) = \mathrm{E}\{X^1(s)\,(X^1(t))'\}, \quad 0 \leqslant s \leqslant t \leqslant 1,$$

whose elements $\gamma_{ij}^1(s, t)$ are given by

$$\gamma_{ij}^1(s, t) = \lambda \sum_{\nu=1}^{M} \mu_\nu \int_0^s \sigma_{ij}^\nu(s-\tau, t-\tau)\,\mathrm{d}\tau.$$

Let $\{X^2(t): 0 \leqslant t \leqslant 1\}$ be a k-dimensional centered Gaussian process expressible in terms of the following stochastic integrals:

$$X_i^2(t) = \sum_{\nu=1}^{M} \int_0^t \mu_i^\nu(t-\tau)\,\mathrm{d}\eta_\nu(\tau), \quad 0 \leqslant t \leqslant 1, \quad i = 1, 2, \ldots, k.$$

With this notation in hand, we proceed to prove the following result on the convergence of the f.d.d.'s of $\{X_n : n \geqslant 1\}$.

Theorem 3.1. *If $\zeta_i^\nu(t) \leqslant C_i < \infty$ for $0 \leqslant t \leqslant 1$, $\nu = 1, 2, \ldots, M$, and $i = 1, 2, \ldots, k$, then the finite dimensional distributions of $\{X_n : n \geqslant 1\}$ converge weakly to those of $X^1 + X^2$, where X^1 and X^2 are independent copies of the centered Gaussian processes described above.*

Proof. It suffices to show convergence of the two-dimensional distributions since the extension to the general case is routine, although tedious. Select time points s and t $(0 \leqslant s \leqslant t \leqslant 1)$ set

$$f_n(u, v\colon s, t) = \mathrm{E}\{\exp[\mathrm{i}(u'X_n(s) + v'X_n(t))]\} \tag{7}$$

for vectors $u, v \in \mathrm{R}^k$. We shall carry out this integration of a bounded function using Fubini's theorem, integrating first with respect to P^2 for each fixed ω^1 and then with respect to P^1. For a fixed ω^1, let $I_n(\omega^1)$ be the integral with respect to P^2 of the expression in braces on the right-hand side of (7). A simple calculation shows that

$$\begin{aligned} I_n(\omega^1) = &\prod_{l=1}^{A_n(s)} \prod_{\nu=1}^{M} [\varphi_n^\nu(u, v; s-\tau_l(n), t-\tau_l(n))]^{m_{l\nu}} \\ &\times \prod_{l=A_n(s)+1}^{A_n(t)} \prod_{\nu=1}^{M} [\varphi_n^\nu(0, v; s-\tau_l(n), t-\tau_l(n))]^{m_{l\nu}} \\ &\times \exp\{-\mathrm{i}n^{1/2}(u'b(s) + v'b(t))\}, \end{aligned}$$

where

$$\varphi_n^\nu(u, v; s, t) = \mathrm{E}\{\exp[\mathrm{i}n^{-1/2}(u'\xi^\nu(s) + v'\xi^\nu(t)]\}.$$

Taking logarithms we have

$$\begin{aligned} \ln I_n(\omega^1) = &\sum_{l=1}^{A_n(s)} \sum_{\nu=1}^{M} m_{l\nu} \ln\varphi_n^\nu(u, v; s-\tau_l(n), t-\tau_l(n)) \\ &+ \sum_{l=A_n(s)+1}^{A_n(t)} \sum_{\nu=1}^{M} m_{l\nu} \ln\varphi_n^\nu(0, v; s-\tau_l(n), t-\tau_l(n)) \\ &- \mathrm{i}n^{1/2}(u'b(s) + v'b(t)). \end{aligned} \tag{8}$$

By ln we shall always mean the determination of the logarithm with angle in $(-\pi, \pi]$.

Since $\zeta_i^\nu(t) \leqslant \max\{C_i^\nu\colon i = 1, ..., k, \nu = 1, ..., M\} = C < \infty$, we can expand the $\ln\varphi_n^\nu$ as follows

$$\begin{aligned} \ln\varphi_n^\nu(u, v; s, t) = &\,\mathrm{i}n^{-1/2}(u'\mu^\nu(s) + v'\mu^\nu(t)) \\ &- \tfrac{1}{2} n^{-1}\{u'\Sigma^\nu(s, s)u + 2u'\Sigma^\nu(s, t)v + v'\Sigma^\nu(t, t)v\} \\ &+ \frac{\theta}{6n^{3/2}} \cdot \frac{6C}{(\frac{1}{2})^3}\left[\sum_{i=1}^{k}(|u_i| + |v_i|)\right]^3, \end{aligned} \tag{9}$$

for n sufficiently large, where θ is a generic complex number with $|\theta| \leqslant 1$. For the estimate of the remainder term see [6, p. 203].

Using (4) and (9) in (8) yields the expression

$$
\begin{aligned}
(10) \qquad \ln I_n(\omega^1) = {} & \mathrm{i} \sum_{\nu=1}^{M} \int_0^s u' \boldsymbol{\mu}^\nu(s-\tau)\, \mathrm{d}T^*_{n\nu}(\tau) \\
& + \mathrm{i} \sum_{\nu=1}^{M} \int_0^t v' \boldsymbol{\mu}^\nu(t-\tau)\, \mathrm{d}T^*_{n\nu}(\tau) \\
& - \tfrac{1}{2} \sum_{\nu=1}^{M} \int_0^s u' \Sigma^\nu(s-\tau, s-\tau) u\, \mathrm{d}T_{n\nu}(\tau) \\
& - \sum_{\nu=1}^{M} \int_0^s u' \Sigma^\nu(s-\tau, t-\tau) v\, \mathrm{d}T_{n\nu}(\tau) \\
& - \tfrac{1}{2} \sum_{\nu=1}^{M} \int_0^t v' \Sigma^\nu(t-\tau, t-\tau) v\, \mathrm{d}T_{n\nu}(\tau) \\
& + 8C\theta \left[\sum_{i=1}^{k} (|u_i| + |v_i|) \right]^3 n^{-1/2} \sum_{\nu=1}^{M} T_{n\nu}(t),
\end{aligned}
$$

for n sufficiently large, uniformly in ω^1. Now let $n \to \infty$. Using Lemma 2.1 and the definition of weak convergence, we see that the last four terms on the right-hand side of (10) converge a.e. to

$$
-\tfrac{1}{2} \lambda \sum_{\nu=1}^{M} \mu_\nu \left\{ \int_0^s \{ u' \Sigma^\nu(s-\tau, s-\tau) u + 2u' \Sigma^\nu(s-\tau, t-\tau) v \}\, \mathrm{d}\tau \right.
$$
$$
\left. + \int_0^t v' \Sigma^\nu(t-\tau, t-\tau) v\, \mathrm{d}\tau \right\}.
$$

Next we consider the first two terms on the right-hand-side of (10). Let $g^1_\nu(\tau) = u' \boldsymbol{\mu}^\nu(\tau)$ and $g^2_\nu(\tau) = v' \boldsymbol{\mu}^\nu(\tau)$ and define the mappings f^i_ν: $(D, \mathcal{D}) \to (R^1, \mathcal{R}^1)$ for $i = 1, 2$, $\nu = 1, \ldots, M$, and $t \in [0, 1]$ by

$$
f^i_\nu(x)(t) = g^i_\nu(0) x(t) - g^i_\nu(t) x(0) + \int_0^t x(t-\tau)\, \mathrm{d}g^i_\nu(\tau).
$$

Now for our fixed s and t let f: $(D^k, \mathcal{D}^k) \to (\mathrm{R}^1, \mathcal{R}^1)$ be given by

$$f(x) = \sum_{\nu=1}^{M} [f_\nu^1(x_\nu)(s) + f_\nu^2(x_\nu)(t)] .$$

Observe that f is continuous at those points $x \in C^k$, since then Skorohod convergence implies uniform convergence (consult [2, p. 112]). Lemma 2.4 shows that $T_n^* \Rightarrow \eta$ and P $\{\eta \in C^k\} = 1$. Thus by the Continuous Mapping Theorem [2, Theorem 5.1], $f(T_n^*) \Rightarrow f(\eta)$ as $n \to \infty$. Recall that the stochastic integral

$$\int_0^t g(t-\tau)\,\mathrm{d}\xi(\tau) = g(0)\,\xi(t) - g(t)\,\xi(0) + \int_0^t \xi(t-\tau)\,\mathrm{d}g(\tau) \text{ a.e.},$$

where ξ is Brownian motion and g is of bounded variation on [0, 1] (see [10, p. 1]). Thus

$$\begin{aligned} f(T_n^*) &\Rightarrow \sum_{\nu=1}^{M} \left\{ \int_0^s u'\mu^\nu(s-\tau)\,\mathrm{d}\eta_\nu(\tau) + \int_0^s v'\mu^\nu(t-\tau)\,\mathrm{d}\eta_\nu(\tau) \right\} \\ &= u'X^2(s) + v'X^2(t). \end{aligned}$$

Since the r.v. I_n is bounded, we have

$$\begin{aligned} \lim_{n\to\infty} f_n(u,v;s,t) &= \mathrm{E}\{\exp[\mathrm{i}(u'X^1(s) + v'X^1(t)]\} \\ &\quad \times \mathrm{E}\{\exp[\mathrm{i}(u'X^2(s) + v'X^2(t)]\}, \end{aligned}$$

which completes the proof.

An examination of the proof shows that the X^1 (X^2) process in the limit is due to the fluctuations of the secondary processes (primary process plus random numbers of each type). If the secondary processes are deterministic ($\Sigma^\nu(s,t) \equiv 0$), then X^1 does not appear in the limit. If, in addition, the $\mu^\nu(t)$ are constant in t, this result is simply the c.l.t. for the random sum of random vectors. On the other hand, if the renewal epochs are deterministic at 1, 2, ..., the $m_{l\nu} \equiv 1$, $M = 1$ and $\xi_{l1}^1(t) = X_l$ for all $t \in [0, 1]$ (where $\{X_l : l \geqslant 1\}$ is a sequence of i.i.d. random vectors), then this result is the convergence of f.d.d.'s of the processes which arise in the vector version of Donsker's Theorem in $D[0, 1]$ (cf. [2, p. 137]).

4. Tightness and weak convergence

To complete the proof that $X_n \Rightarrow X^1 + X^2$, we must show that the sequence of processes $\{X_n : n \geqslant 1\}$ is tight. However, before doing this, we first deal with tightness in $D[0, 1]$ in general. The fact that a sequence of processes is tight essentially guards against extremely wild fluctuations in small intervals of time. When working with applications, the easiest conditions to check for tightness are those expressed in terms of product moments. For convenience in exposition we state the following slight variant of [2, Theorem 15.6 and p. 133]. To that end, let $\{Z_n : n \geqslant 1\}$ and Z be elements of D and let T_Z consist of those $t \in [0, 1]$ for which

$$\mathbf{P}\{Z(t) \neq Z(t-)\} = 0.$$

Recall that the points 0 and 1 always belong to T_Z.

Lemma 4.1. *Suppose that the following conditions hold*:

(i) $(Z_n(t_1), ..., Z_n(t_k)) \Rightarrow (Z(t_1), ..., Z(t_k))$ *whenever* $t_1, ..., t_k$ *all belong to* T_Z.

(ii) $\mathbf{P}\{Z(1) \neq Z(1-)\} = 0$.

(iii) *There exists an integer* n_0 *such that*

$$\mathbf{E}\{|Z_n(t)-Z_n(s)|^\gamma |Z_n(u)-Z_n(t)|^\gamma\}$$

$$\leqslant [F_n(t)-F_n(s)]^\alpha [F_n(u)-F_n(t)]^\alpha$$

for $0 \leqslant s \leqslant t \leqslant u \leqslant 1$ *and* $n \geqslant n_0$, *where* $\gamma \geqslant 0$, $\alpha > \frac{1}{2}$ *and the* F_n $(n \geqslant n_0)$ *are nondecreasing functions on* $[0, 1]$ *which converge pointwise to F, a nondecreasing right-continuous function on* $[0, 1]$.

Then $Z_n \Rightarrow Z$.

The observation that the function on the right-hand side of the inequality in (iii) can be allowed to depend on n was made by Bickel and Wichura [1, p. 1666].

We return now to our c.s.p.'s $\{X_n : n \geqslant 1\}$. First consider the case $k = M = 1$ and let $m_{l1} = m_l$, $\mathrm{E}\{m_l\} = \mu$ and $T_n(t) = n^{-1} \sum_{\nu=1}^{A_n(t)} m_l$. In view of condition (iii) of Lemma 4.1, it is natural to hope that a product moment assumption of this type on the secondary processes will imply the same thing for the compound process. In other words, the superposition of secondary processes with mild fluctuations will not result in fluctuations of the compound process large enough to destroy tightness. This hope is borne out in the next lemma. We let K denote a generic positive constant, not always the same.

Lemma 4.2. *Suppose the following conditions hold:*

(i) $\mu(t) = 0$ *for all* $t \in [0, 1]$.

(ii) $\mathrm{P}\{\xi(0) = 0\} = 1$.

(iii) $\mathrm{E}\{[\xi(t)-\xi(s)]^2\} \leqslant K(t-s)^\alpha$ *for all* $0 \leqslant s \leqslant t \leqslant 1$, *where* $\alpha = \frac{1}{2}$.

(iv) $\mathrm{E}\{[\xi(t)-\xi(s)]^2\,[\xi(u)-\xi(t)]^2\} \leqslant K(t-s)^\alpha (u-t)^\alpha$ *for all* $0 \leqslant s \leqslant t \leqslant u \leqslant 1$.

(v) $\mathrm{E}\{m_1^4\} < \infty$.

Then condition (iii) *of Lemma* 4.1 *holds for the process* X_n *with* $\gamma = 2$, $F_n(t) = \{12K\mathrm{E}\{T_n^2(1)\} + n^{-1}K\mathrm{E}\{T_n(1)\}\}^{1/2\alpha}t$, *and* $F(t) = Kt$.

Proof. For a fixed value of ω^1, we compute the expected value of $[X_n(t)-X_n(s)]^2\,[X_n(u)-X_n(t)]^2$ with respect to P^2 and call it $J_n(\omega^1)$. Using (i)–(iv) and the independence of the secondary processes, we obtain, after a lengthy computation involving liberal use of the Cauchy-Schwarz inequality, the following inequality:

$$J_n \leqslant \{12KT_n^2(1) + n^{-1}KT_n(1)\}\,(t-s)^\alpha(u-t)^\alpha\,. \tag{11}$$

Next taking the expectation with respect to P^1, we have

$$\mathrm{E}\{[X_n(t)-X_n(s)]^2\,[X_n(u)-X_n(t)]^2\}$$
$$\leqslant (F_n(t)-F_n(s))^\alpha(F_n(u)-F_n(t))^\alpha.$$

Now use condition (v) and the method of Lemma 2.2 to conclude that $\mathrm{E}\{T_n^2(1)\} \to (\lambda\mu)^2$. Of course $n^{-1}\mathrm{E}\{T_n(1)\} \to 0$. Hence $F_n(1) \to F(t)$ and the proof is complete.

It should be possible to generalize conditions (iii) and (iv) of Lemma 4.2 to permit t, say, to be replaced by $G(t)$, where G is a non-decreasing, right-continuous function on $[0, 1]$. However, this generalization requires many additional technical details that do not add to our understanding of the situation. Furthermore, the stated conditions (iii) and (iv) are satisfied by all our examples.

We have assumed that our secondary processes exist as random elements in D. It should be remarked that condition (iii) of Lemma 4.2 (together with the consistency of the f.d.d.'s) implies that ξ exists as a random element of D; see [2, Theorem 15.7 and eq. (15.44)].

Now we can proceed to the proof of weak convergence of the sequence $\{X_n : n \geqslant 1\}$. For convenience, let $\boldsymbol{\xi}^\nu$ denote a generic secondary process of type ν and set $\boldsymbol{\beta}^\nu = \boldsymbol{\xi}^\nu - \boldsymbol{\mu}^\nu$.

Theorem 4.3. *Suppose that the following conditions hold:*

(i) $\zeta_i^\nu(t) \leqslant C_i^\nu < \infty$.

(ii) $\mathrm{P}\{\beta_i^\nu(0) = 0\} = 1$.

(iii) $\mathrm{E}\{[\beta_i^\nu(t) - \beta_i^\nu(s)]^2\} \leqslant K(t-s)^\alpha$.

(iv) $\mathrm{E}\{[\beta_i^\nu(t) - \beta_i^\nu(s)]^2 [\beta_i^\nu(u) - \beta_i^\nu(t)]^2\} \leqslant K(t-s)^\alpha (u-t)^\alpha$.

(v) $\mathrm{E}\{m_{1\nu}^4\} < \infty$, *where* $\alpha > \frac{1}{2}$ *and* (i)–(v) *are to hold for all* $0 \leqslant s \leqslant t \leqslant u \leqslant 1$, $\nu = 1, ..., M$, *and* $i = 1, ..., k$.

Then $X_n \Rightarrow X^1 + X^2$.

Proof. We have shown in Theorem 3.1 under condition (i) that the f.d.d.'s of X_n converge to those of $X^1 + X^2$. By Lemma 2.6, it will suffice to show that the sequences $\{X_{ni} : n \geqslant 1\}$, $i = 1, ..., k$, are tight. We can write

$$X_{ni}(t) = \sum_{\nu=1}^{M} X_{ni}^\nu(t),$$

where

$$X_{ni}^\nu(t) = n^{-1/2} \left[\sum_{l=1}^{A_n(t)} \sum_{j=1}^{m_{l\nu}} \xi_{lji}^\nu(t - \tau_l(n)) - n\lambda\mu_\nu \int_0^t \mu_i^\nu(\tau)\mathrm{d}\tau \right].$$

Next write $X_{ni}^\nu(t)$ as

$$(12) \qquad n^{-1/2} \left[\sum_{l=1}^{A_n(t)} \sum_{j=1}^{m_{l\nu}} \beta_{lji}^\nu(t - \tau_l(n)) \right]$$

$$+ \int_0^t \mu_i(t-\tau)\mathrm{d}T_{n\nu}^*(\tau) = D_n(t) + E_n(t).$$

Using Lemma 2.4, the continuous mapping theorem and the argument used in Theorem 3.1, we have

$$E_n \Rightarrow E,$$

where

$$E(t) = \int_0^t \mu_i^\nu(t-\tau)\,\mathrm{d}\eta_\nu(\tau)$$

with $\mathbf{P}\{E \in C\} = 1$. Hence by Lemma 2.5 for each $\epsilon, \eta > 0$ there exists δ $(0 < \delta < 1)$ and n_0 such that

$$\mathbf{P}\{w_{E_n}(\delta) \geqslant \epsilon\} \leqslant \eta \quad \text{for} \quad n \geqslant n_0. \tag{13}$$

Using the same method employed in Theorem 3.1, we can show that the f.d.d.'s of D_n converge to those of D, where $\mathbf{P}\{D \in C\} = 1$. Conditions (ii) and (iii) imply, using Lemma 4.2, that condition (iii) of Lemma 4.1 holds. Hence $D_n \Rightarrow D$ and by Prohorov's theorem (cf. [2, Theorem 6.2]) $\{D_n : n \geqslant 1\}$ is tight. By Lemma 2.5, again (12) holds for D_n. Since

$$\mathbf{P}\{w_{X_{ni}^\nu}(\delta) \geqslant \epsilon\} \leqslant \mathbf{P}\{w_{D_n}(\delta) \geqslant \tfrac{1}{2}\epsilon\} + \mathbf{P}\{w_{E_n}(\delta) \geqslant \tfrac{1}{2}\epsilon\},$$

(13) also holds for X_{ni}^ν and by the same argument for X_{ni}. Finally, since $X_{ni}(0) = 0$, $\{X_{ni}: n \geqslant 1\}$ is tight by [2, Theorem 15.5].

The next corollary follows immediately from Theorem 5.5 of [2] by taking $h_n(x) = n^{-1/2}x$ and $h(x) = 0$.

Corollary 4.4. *If the hypotheses of Theorem 5.5 hold, then as $n \to \infty$, $n^{-1} Y_n \Rightarrow b$.*

5. Applications

In this section, we apply the weak convergence result of Theorem 4.3 to a number of compound stochastic processes with specific secondary processes.

5.1. *An urn model with immigration.* This example uses the full generality of our c.s.p. Consider a k-urn model, $k > 1$, into which balls immigrate. Initially the urns are empty, but at the epochs of a renewal process, a random number of balls is placed in each urn. Once the balls are in the urns, they move (migrate) independently from urn to urn according to a k-state continuous parameter Markov chain (M.c.). The immigration and migration mechanisms are assumed to be independent. Assume that the M.c. has transition probabilities $\{p_{ij}(t): t \geqslant 0, i, j = 1, \ldots, k\}$. Let the ν^{th} secondary process have components $I_{\{\tau: Y^\nu(\tau)=i\}}$, where Y^ν is the M.c. starting in state ν and I_A is the indicator function of the set A. Hence $\xi^\nu(t)$ is a vector with all components zero except for a one in the component corresponding to the state of the M.c. at time t. Thus $Y(t)$ is the random vector describing the number of balls in the various urns at time t. It is easy to check that $\mu_i(t) = p_{\nu i}(t)$, $\sigma^\nu_{ij}(s, t) = p_{\nu i}(s) \times [p_{ij}(t-s) - p_{\nu j}(t)]$, and $\zeta^\nu_i(t) \leqslant 1$. Using the inequalities

$$
(14) \qquad \begin{aligned} |p_{ij}(t) - p_{ij}(s)| &\leqslant q_i(t-s), \\ 1 - p_{ii}(t) &\leqslant q_i t, \end{aligned}
$$

which are valid for all i, j, and $0 \leqslant s \leqslant t$ (cf. [4, p. 130]), we can show that

$$
\mathbf{E}\{[\beta^\nu_i(t) - \beta^\nu_i(s)]^2\} \leqslant K(t-s)
$$

and

$$
\mathbf{E}\{[\beta^\nu_i(t) - \beta^\nu_i(s)]^2 [\beta^\nu_i(u) - \beta^\nu_i(t)]^2\} \leqslant K^2(t-s)(u-t),
$$

where $0 < K < \infty$. The covariance of X^1 is

$$
\gamma_{ij}(s, t) = \sum_{\nu=1}^{M} \int_0^s p_{\nu i}(s-\tau)\,[p_{ij}(t-s-\tau) - p_{\nu j}(t-\tau)]\,\mathrm{d}\tau
$$

and

$$
X^2_i(t) = \sum_{\nu=1}^{M} \int_0^t p_{\nu i}(t-\tau)\,\mathrm{d}\eta_\nu(\tau).
$$

Theorem 4.3 is satisfied with $\alpha = 1$ and hence $X_n \Rightarrow X^1 + X^2$.

At the expense of making some of the calculations more difficult, we could assume that the migration process is governed by a semi-Markov chain. A model similar to this one was studied by Port [12; 13]. This urn model has a great number of applications. Essentially, any time one has people or objects entering a system at random epochs and moving to different classes according to a M.c. this model holds. If one of the urns is considered an absorbing state in the M.c., then the model can accommodate departures from the system as well as arrivals. Hospitals, judicial systems, personnel systems, k-lane freeways, and university faculties are among examples of such systems.

5.2. *The $GI/G/\infty$ queue.* Consider the queueing problem in which customers of M types arrive as in our c.s.p., a customer of type ν has positive service time υ^ν with d.f. H^ν, and there are an infinite number of servers. We assume that the continuous part of H^ν has a continuous density and satisfies $H^\nu(t)-H^\nu(s) \leqslant K(t-s)^\alpha$, where $\alpha > \frac{1}{2}$. Let the secondary process of type ν be given by $\xi^\nu = I_{\{0 \leqslant s < \upsilon^\nu\}}$. Then $k = 1$ and $Y(t)$ is simply the number of customers in the system at time t.

This example also has an interpretation as a type II counter. We can easily show that

$$\mu^\nu(t) = \mathrm{P}\{t < \upsilon^\nu\} = 1 - H^\nu(t),$$

$$\sigma^\nu(s, t) = H^\nu(s)\,[1 - H^\nu(t)],$$

$$\mathrm{P}\{\beta^\nu(0) = 0\} = 1,$$

$$\zeta^\nu(t) \leqslant 1,$$

$$\mathrm{E}\{[\beta^\nu(t) - \beta^\nu(s)]^2\} \leqslant H^\nu(t) - H^\nu(s),$$

$$\mathrm{E}\{[\beta^\nu(t) - \beta^\nu(s)]^2\,[\beta^\nu(u) - \beta^\nu(t)]^2\} \leqslant 2[H^\nu(t) - H^\nu(s)] \times [H^\nu(u) - H^\nu(t)]$$

for $0 \leqslant s \leqslant t \leqslant u \leqslant 1$. Thus X^1 has covariance function

$$\gamma^1(s, t) = \lambda \sum_{\nu=1}^{M} \int_0^s H^\nu(s-\tau)\,[1 - H^\nu(t-\tau)]\,\mathrm{d}\tau$$

and

$$X^2(t) = \sum_{\nu=1}^{M} \int_0^t [1 - H^\nu(t-\tau)]\,\mathrm{d}\eta_\nu(\tau).$$

Again Theorem 4.3 is satisfied and hence $X_n \Rightarrow X^1 + X^2$.

5.3. *Shot noise.* A well-known example of a c.p.s. is the current generated in a vacuum tube as electrons pass from cathode to anode. The c.p.s. $Y(t)$ represents the current at time t when the secondary process is given by $\xi(t) = X\mathrm{e}^{-at}$, where a is a positive constant and X is a r.v. with finite fourth moment. In this case $k = M = 1$ and $m_{l1} \equiv 1$. Then

$$\mu(t) = \mathrm{E}\{X\}\mathrm{e}^{-at},$$

$$\sigma(s, t) = \sigma^2\{X\}\mathrm{e}^{-a(s+t)},$$

$$\zeta(t) \leqslant \mathrm{E}\{|X|^3\} < \infty,$$

$$\mathrm{E}\{[\beta(t) - \beta(s)]^2\} \leqslant \sigma^2\{X\}[B(t) - B(s)],$$

$$\mathrm{E}\{[\beta(t) - \beta(s)]^2\,[\beta(u) - \beta(t)]^2\}$$

$$\leqslant \mathrm{E}\{(X - \mathrm{E}\{X\})^4\}\,[B(t) - B(s)]\,[B(u) - B(t)],$$

where $B(t) = 1 - \mathrm{e}^{-at}$. Of course, $B(t) - B(s) \leqslant a(t-s)$. The covariance of X^1 is

$$\sigma_1(s, t) = \lambda\sigma^2\{X\}\int_0^s \mathrm{e}^{-a(s+t-\tau)}\mathrm{d}\tau$$

$$= \tfrac{1}{2}a^{-1}\,\lambda\sigma^2\{X\}\,(\mathrm{e}^{2as} - 1)\mathrm{e}^{-a(s+t)},$$

and

$$X^2(t) = \sigma\{u_1\}\,\lambda^{3/2}\,\mathrm{E}\{X\}\int_0^t \mathrm{e}^{-a(t-\tau)}\mathrm{d}\eta^*(\tau),$$

where η^* is standard one-dimensional Brownian motion. Theorem 4.3 is satisfied with $\alpha = 1$ and hence $X_n \Rightarrow X^1 + X^2$.

5.4. *Brownian motion.* For simplicity we take $k = M = 1$ and the secondary process ξ to be standard Brownian motion on $[0, 1]$. Thus $Y(t)$ is the sum of the displacements of Brownian travelers beginning their sojourns at random epochs. Recall that

$$\mu(t) = 0,$$

$$\sigma(s,t) = \mathrm{E}\{\xi(s)\xi(t)\} = s,$$

$$\zeta(t) = \mathrm{E}\{|\xi(t)|^3\} = 4(2\pi)^{-1/2}t^{3/2},$$

$$\mathrm{E}\{[\xi(t)-\xi(s)]^2\} = t-s,$$

$$\mathrm{E}\{[\xi(u)-\xi(t)]^2[\xi(t)-\xi(s)]^2\} = (t-s)(u-t)$$

for $0 \leqslant s \leqslant t \leqslant u \leqslant 1$. Hence $X^2 \equiv 0$ and X^1 has covariance function $\gamma^1(s,t)$ given by

$$\gamma^1(s,t) = \lambda\mu_1 \int_0^s (s-\tau)\,\mathrm{d}\tau = \tfrac{1}{2}\lambda\mu_1 s^2 .$$

The conditions of Theorem 4.3 are clearly satisfied with $\alpha = 1$. Hence $X_n \Rightarrow X^1$.

This example could easily be generalized to permit ξ to be Brownian motion with drift (or even more general Gaussian processes) and k, $M > 1$.

5.5. *Brownian bridge*. Again take $k = M = 1$ and let the secondary process ξ^0 be Brownian bridge or the so-called tied-down Brownian motion. Recall that ξ^0 can be constructed from Brownian motion ξ by taking $\xi^0(t) = \xi(t) - t\xi(1)$ for $0 \leqslant t \leqslant 1$ (cf. [2, p. 65]). Then

$$\mu(t) = 0,$$

$$\sigma(s,t) = s(1-t),$$

$$\zeta(t) = 4(2\pi)^{-1/2}[t-(1-t)]^{3/2},$$

$$\mathrm{E}\{[\xi^0(t)-\xi^0(s)]^2\} = (t-s)(1-(t-s)) \leqslant t-s$$

for $0 \leqslant s \leqslant t \leqslant 1$. A tedious calculation shows that

$$\mathrm{E}\{[\xi^0(t)-\xi^0(s)]^4\} \leqslant 14(t-s)^2 ;$$

hence, using Cauchy-Schwarz,

$$E\{[\xi^0(t)-\xi^0(s)]^2\,[\xi^0(u)-\xi^0(t)]^2\}$$
$$\leqslant E^{1/2}\{[\xi^0(t)-\xi^0(s)]^4\} \times E^{1/2}\{[\xi^0(u)-\xi^0(s)]^4\}$$
$$\leqslant 14(t-s)(u-t)$$

for $0 \leqslant s \leqslant t \leqslant 1$. Again $X^2 \equiv 0$, and now X^1 has covariance function

$$\gamma^1(s,t) = \lambda\mu_1 \int_0^s (s-\tau)(1-t+\tau)\,\mathrm{d}\tau = \lambda\mu_1\,[\tfrac{1}{2}s^2(1-t)+\tfrac{1}{6}s^3].$$

The conditions of Theorem 4.3 are satisfied again with $\alpha = 1$, so $X_n \Rightarrow X^1$

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